On a weak Gauß law in general relativity and torsion

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Abstract

We present an explicit example showing that the weak Gauß law of general relativity (with cosmological constant) fails in Einstein-Cartan's theory. We take this as an indication that torsion might replace dark matter.

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1 Introduction

Gauß' law in Maxwell's theory allows to link the total charge of an isolated charge distribution to the asymptotic strength of its electric field. In general relativity a weak form of Gauß' law holds in the static and spherically symmetric case and remains true when a cosmological constant is added. We show that this weak Gauß law fails when torsion is added à la Einstein-Cartan [1, 2, 3].

2 The weak Gauß law of general relativity

To warm up, let us quickly review the derivation of the weak Gauß law in general relativity with cosmological constant.

In a first step we solve the Killing equation

$$\xi^\alpha \frac{\partial}{\partial x^\alpha} g_{\mu\nu} + \frac{\partial \xi^{\bar\mu}}{\partial x^\mu} g_{\bar\mu\nu} + \frac{\partial \xi^{\bar\nu}}{\partial x^\nu} g_{\mu\bar\nu} = 0. \tag{1}$$

in the static, spherical case, i.e. for the four vector fields: $\xi = \partial/\partial t$ generating time translation and

$$\xi = -\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\varphi}, \tag{2}$$

$$\xi = +\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\varphi}, \tag{3}$$

$$\xi = \frac{\partial}{\partial\varphi}, \tag{4}$$

generating the rotations around the x-, y- and z-axis. In fact, since for the infinitesimal rotations, the commutator of the last two gives the first, we may discard the infinitesimal rotation around the x-axis. It is a typical accountant's work to list and solve the remaining 3×10 linear, first order partial differential equations for the 10 unknowns $g_{\mu\nu}$. The solution is well known:

$$\mathrm{d}\tau^2 = B\,\mathrm{d}t^2 + 2D\,\mathrm{d}t\,\mathrm{d}r - A\,\mathrm{d}r^2 - C\,\mathrm{d}\theta^2 - C\,\sin^2\theta\,\mathrm{d}\varphi^2, \tag{5}$$

with four functions A, B, C and D of r. By a suitable coordinate transformation we may achieve $D = 0$ and $C = r^2$. Then A and B are positive.

In a second step we solve the Einstein equation

$$\mathrm{Ricci}_{\mu\nu} - \tfrac{1}{2}\,\mathrm{scalar}\, g_{\mu\nu} - \Lambda\, g_{\mu\nu} = 8\pi G\, \tau_{\mu\nu}, \tag{6}$$

in the vacuum, $\tau_{\mu\nu} = 0$, and obtain the Kottler or Schwarzschild - de Sitter solution:

$$B = \frac{1}{A} = 1 - \frac{S}{r} - \tfrac{1}{3}\Lambda\, r^2, \tag{7}$$

with an integration constant S.

In a third step we use the weak Gauß law to determine the integration constant S in terms of the total mass of a spherically symmetric mass distribution ${\tau^t}_t(r)$ with support inside a ball of radius R (not to be confused with the curvature scalar, that we denote by 'scalar'). This law is simply the ${}^t{}_t$ component of the Einstein equation:

$$\mathrm{Ricci}^t{}_t - \tfrac{1}{2}\,\mathrm{scalar}\, g^t{}_t - \Lambda\, g^t{}_t = 8\pi G\, \tau^t{}_t\,. \tag{8}$$

In the static, spherical case this equation reduces to:

$$\frac{1}{r^2}\left[1 - \frac{\mathrm{d}}{\mathrm{d}r}\frac{r}{A}\right] - \Lambda = 8\pi G\, \tau^t{}_t\,. \tag{9}$$

Solving for $\mathrm{d}(r/A)/\mathrm{d}r$ and integrating we have:

$$\frac{r}{A} = r - 2G\int_0^r \tau^t{}_t(\tilde r)\, 4\pi\tilde r^2\, \mathrm{d}\tilde r - \tfrac{1}{3}\Lambda\, r^3 + K. \tag{10}$$

The integration constant K is seen to vanish by evaluating at $r = 0$ and noting that $A(0)$ is positive. Then the interior solution is

$$A(r) = \left[1 - \left(2G\int_0^r \tau^t{}_t(\tilde r)\, 4\pi\tilde r^2\, \mathrm{d}\tilde r\right)/r - \tfrac{1}{3}\Lambda\, r^2\right]^{-1}, \tag{11}$$

and from the continuity of A at $r = R$ we obtain the Schwarzschild radius $S = 2GM$ with

$$M := \int_0^R \tau^t{}_t(\tilde r)\, 4\pi\tilde r^2\, \mathrm{d}\tilde r. \tag{12}$$

3 The weak Gauß law in Einstein-Cartan's theory

Our task is to redo the above three steps including torsion à la Einstein-Cartan.

3.1 First step: invariant connection

We have to solve the analogue of the Killing equation [4] for the now independent (metric) connection Γ

$$\xi^\alpha \frac{\partial}{\partial x^\alpha}\Gamma^\lambda{}_{\mu\nu} - \frac{\partial \xi^\lambda}{\partial x^{\bar\lambda}}\Gamma^{\bar\lambda}{}_{\mu\nu} + \frac{\partial \xi^{\bar\mu}}{\partial x^\mu}\Gamma^\lambda{}_{\bar\mu\nu} + \frac{\partial \xi^{\bar\nu}}{\partial x^\nu}\Gamma^\lambda{}_{\mu\bar\nu} + \frac{\partial^2\xi^\lambda}{\partial x^\mu \partial x^\nu} = 0, \tag{13}$$

for the vector fields generating time translation and rotations. These 3×64 equations yield the following non-vanishing connection components:

$$\Gamma^a{}_{bc} = X^a{}_{bc}(r), \qquad a,\, b,\, c \in \{t, r\}, \tag{14}$$
$$\Gamma^a{}_{\theta\theta} = \Gamma^a{}_{\varphi\varphi}/\sin^2\theta = E^a(r), \qquad \Gamma^a{}_{\theta\varphi} = -\Gamma^a{}_{\varphi\theta} = \sin\theta\, F^a(r), \tag{15}$$
$$\Gamma^\theta{}_{a\theta} = \Gamma^\varphi{}_{a\varphi} = C_a(r), \qquad \Gamma^\theta{}_{\theta a} = \Gamma^\varphi{}_{\varphi a} = Y_a(r), \tag{16}$$
$$\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\,\cos\theta, \qquad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = \cos\theta/\sin\theta, \tag{17}$$

with $8+2+2+2+2=16$ arbitrary functions X, E, F, C, Y of r. The metricity condition,

$$\frac{\partial}{\partial x^\lambda}\, g_{\mu\nu} - \Gamma^{\bar\mu}{}_{\mu\lambda} g_{\bar\mu\nu} - \Gamma^{\bar\nu}{}_{\nu\lambda} g_{\mu\bar\nu} = 0, \tag{18}$$

reduces the 16 functions to four arbitrary functions C_t, C_r, D_t, D_r and we have the following non-vanishing connection components:

$$\Gamma^t{}_{tr} = \tfrac{1}{2} B'/B, \quad \Gamma^t{}_{ra} = D_a\, A/B, \qquad \Gamma^r{}_{rr} = \tfrac{1}{2} A'/A, \quad \Gamma^r{}_{ta} = D_a, \tag{19}$$

$$\Gamma^t{}_{\theta\theta} = \Gamma^t{}_{\varphi\varphi}/\sin^2\theta = C_t\, r^2/B, \qquad \Gamma^r{}_{\theta\theta} = \Gamma^r{}_{\varphi\varphi}/\sin^2\theta = -C_r\, r^2/A, \tag{20}$$

$$\Gamma^\theta{}_{a\theta} = \Gamma^\varphi{}_{a\varphi} = C_a, \qquad \Gamma^\theta{}_{\theta r} = \Gamma^\varphi{}_{\varphi r} = 1/r, \tag{21}$$

$$\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\,\cos\theta, \qquad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = \cos\theta/\sin\theta. \tag{22}$$

3.2 Einstein's equation

So far we have used a holonomic frame $\mathrm{d}x^\mu$. Now it will be convenient to work in an orthornormal frame $e^a =: e^a{}_\mu\, \mathrm{d}x^\mu$. We will use the notations of reference [5]. The metric tensor reads $g_{\mu\nu}(x) = e^a{}_\mu(x)\, e^b{}_\nu(x)\, \eta_{ab}$.

The connection with respect to a holonomic frame is written as a $g\ell(4)$-valued 1-form $\Gamma^\alpha{}_\beta =: \Gamma^\alpha{}_{\beta\mu}\, \mathrm{d}x^\mu$. The link between the components of the same connection with respect to the holonomic frame Γ and with respect to the orthonormal frame ω is given by the $GL(4)$ gauge transformation with $e(x) = e^a{}_\mu(x) \in GL(4)$;

$$\omega = e\Gamma e^{-1} + e\mathrm{d}e^{-1}, \tag{23}$$

or with indices:

$$\omega^a{}_{b\mu} = e^a{}_\alpha\, \Gamma^\alpha{}_{\beta\mu}\, e^{-1\,\beta}{}_b + e^a{}_\alpha\, \frac{\partial}{\partial x^\mu}\, e^{-1\,\alpha}{}_b. \tag{24}$$

For ω the metricity condition is algebraic and means that its values $\omega^a{}_b$ are in the Lie algebra of the Lorentz group: $\omega_{ab} = -\omega_{ba}$.

In the orthonormal frame with $e = \mathrm{diag}(\sqrt{B},\, \sqrt{A},\, r,\, r\sin\theta)$, the non-vanishing components $\omega^a{}_{b\mu}$ of the invariant connection (19–22) are:

$$\omega^t{}_{rt} = D_t\sqrt{A/B}, \quad \omega^t{}_{rr} = D_r\sqrt{A/B}, \quad \omega^t{}_{\theta\theta} = \omega^t{}_{\varphi\varphi}/\sin\theta = C_t\, r/\sqrt{B}, \tag{25}$$

$$\omega^\theta{}_{\varphi\varphi} = -\cos\theta, \qquad \omega^r{}_{\theta\theta} = \omega^r{}_{\varphi\varphi}/\sin\theta = -C_r\, r/\sqrt{A}. \tag{26}$$

The curvature as $so(1,3)$-valued 2-form,

$$R := \mathrm{d}\omega + \tfrac{1}{2}[\omega,\omega], \tag{27}$$

has the following non-vanishing components, $R^a{}_b =: \frac{1}{2} R^a{}_{b\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$:

$$\begin{aligned}
R^t{}_{rtr} = -(D_t\sqrt{A/B})', \quad R^t{}_{\theta t\theta} &= -C_r D_t\, r/\sqrt{B}, & R^t{}_{\theta r\theta} &= (C_t\, r/\sqrt{B})' - C_r D_r\, r/\sqrt{B},\\
R^t{}_{\varphi t\varphi} &= \sin\theta\, R^t{}_{\theta t\theta}, & R^t{}_{\varphi r\varphi} &= \sin\theta\, R^t{}_{\theta r\theta},\\
R^r{}_{\theta t\theta} &= C_t D_t\, r\sqrt{A}/B, & R^r{}_{\theta r\theta} &= -(C_r\, r/\sqrt{A})' + C_t D_r\, r\sqrt{A}/B,\\
R^r{}_{\varphi t\varphi} &= \sin\theta\, R^r{}_{\theta t\theta}, & R^r{}_{\varphi r\varphi} &= \sin\theta\, R^r{}_{\theta r\theta},\\
& & R^\theta{}_{\varphi\theta\varphi} &= \sin\theta\,(1 + C_t^2\, r^2/B - C_r^2\, r^2/A).
\end{aligned} \tag{28}$$

Next we compute the Ricci tensor,

$$\mathrm{Ricci}^a{}_b := \eta^{aa'} R^c{}_{a'\mu\nu} e^{-1\mu}{}_c e^{-1\nu}{}_b, \tag{29}$$

whose non-vanishing components are:

$$\begin{aligned}
\mathrm{Ricci}^t{}_t &= (D_t\sqrt{A/B})'/\sqrt{AB} + 2\,C_r D_t/B, && (30)\\
\mathrm{Ricci}^r{}_r &= (D_t\sqrt{A/B})'/\sqrt{AB} + 2\,(r\,C_r/\sqrt{A})'/(r\sqrt{A}) - 2\,C_t D_r/B, && (31)\\
\mathrm{Ricci}^t{}_r &= -2\,(C_t/\sqrt{B})'/\sqrt{A} - 2\,C_t/(r\sqrt{AB}) + 2\,C_r D_r/\sqrt{AB}, && (32)\\
\mathrm{Ricci}^r{}_t &= -2\,C_t D_t\sqrt{A/B}/B, && (33)\\
\mathrm{Ricci}^\theta{}_\theta &= \mathrm{Ricci}^\varphi{}_\varphi && (34)\\
&= (C_r/\sqrt{A})'/\sqrt{A} - 1/r^2 + C_r/(rA) - C_t^2/B + C_r^2/A - C_t D_r/B + C_r D_t/B.
\end{aligned}$$

The curvature scalar is

$$\begin{aligned}
\mathrm{scalar} &= \mathrm{Ricci}^t{}_t + \mathrm{Ricci}^r{}_r + 2\,\mathrm{Ricci}^\theta{}_\theta\\
&= 2\,(D_t\sqrt{A/B})'/\sqrt{AB} + 4\,(C_r/\sqrt{A})'/\sqrt{A} - 2/r^2 + 4\,C_r/(rA)\\
&\quad -2\,C_t^2/B + 2\,C_r^2/A - 4\,C_t D_r/B + 4\,C_r D_t/B. \qquad (35)
\end{aligned}$$

In an orthonormal frame the Einstein equation reads

$$R^a{}_b - \tfrac{1}{2}\,\mathrm{scalar}\,\delta^a{}_b - \Lambda\,\delta^a{}_b = 8\pi G\,\tau^a{}_b. \tag{36}$$

In our orthonormal frame $e = \mathrm{diag}(\sqrt{B},\ \sqrt{A},\ r,\ r\sin\theta)$, the energy momentum tensor τ_{ab} is symmetric and

(37)

$$\tau^a{}_b = \begin{pmatrix} \rho(r) & q(r) & 0 & 0\\ -q(r) & -p_r(r) & 0 & 0\\ 0 & 0 & -p_\mathrm{a}(r) & 0\\ 0 & 0 & 0 & -p_\mathrm{a}(r)\end{pmatrix}. \tag{38}$$

The tt, rr and $\theta\theta$ components of the Einstein equation are:

$$\begin{aligned}
&-2\,(C_r/\sqrt{A})'/\sqrt{A} + 1/r^2 - 2\,C_r/(rA)\\
&\qquad + C_t^2/B - C_r^2/A + 2\,C_t D_r/B - \Lambda = 8\pi G\,\rho, && (39)\\
&1/r^2 + C_t^2/B - C_r^2/A - 2\,C_r D_t/B - \Lambda = -8\pi G\,p_r, && (40)\\
&-(D_t\sqrt{A/B})'/\sqrt{AB} - (C_r/\sqrt{A})'/\sqrt{A} - C_r/(rA)\\
&\qquad + C_t D_r/B - C_r D_t/B - \Lambda = -8\pi G\,p_\mathrm{a}. && (41)
\end{aligned}$$

The two off-diagonal components read

$$-2\,(C_t/\sqrt{B})'/\sqrt{A} - 2\,C_t/(r\sqrt{AB}) + 2\,C_r D_r/\sqrt{AB} = 2\,C_t D_t\sqrt{A/B}/B = 8\pi G\,q. \tag{42}$$

3.3 Cartan's equation

The torsion as $\mathbb{R}^4$-valued 2-form,

$$T := \mathrm{D}e = \mathrm{d}e + \omega e, \tag{43}$$

has the following non-vanishing components, $T^a =: T_{a'bc}\eta^{a'a}e^b e^c$,

$$T_{ttr} = (D_t - \tfrac{1}{2}B'/A)\,\sqrt{A}/B, \qquad T_{rtr} = D_r/\sqrt{B}, \tag{44}$$

$$T_{\theta t\theta} = T_{\varphi t\varphi} = C_t/\sqrt{B}, \qquad T_{\theta r\theta} = T_{\varphi r\varphi} = (C_r - 1/r)/\sqrt{A}. \tag{45}$$

The Cartan equation,

$$T^c e^d \epsilon_{abcd} = -8\pi G\, s_{ab}, \tag{46}$$

determines the torsion in terms of its source, the half-integer spin current. This is the Lorentz-valued 3-form s_{ab} i.e. the variation of the matter Lagrangian with respect to the spin connection ω_{ab}. To simplify the Cartan equation, let us decompose the torsion tensor into its three irreducible parts:

$$T_{abc} = A_{abc} + \eta_{ab}V_c - \eta_{ac}V_b + M_{abc}, \tag{47}$$

with the completely antisymmetric part $A_{abc} := \frac{1}{3}(T_{abc} + T_{cab} + T_{bca})$, the vector part $V_c := \frac{1}{3}T_{abc}\eta^{ab}$, and the mixed part M_{abc} characterized by $M_{abc} = -M_{acb}$, $M_{abc}\eta^{ab} = 0$, and $M_{abc} + M_{cab} + M_{bca} = 0$. Likewise, we decompose the spin tensor s_{abc} defined by $*s_{ab} =: s_{abc}e^c$;

$$s_{abc} = a_{abc} + \eta_{ca}s_b - \eta_{cb}s_a + m_{abc}, \tag{48}$$

with the completely antisymmetric part $a_{abc} := \frac{1}{3}(s_{abc} + s_{cab} + s_{bca})$, the vector part $s_b := \frac{1}{3}s_{abc}\eta^{ac}$, and the mixed part m_{abc} characterized by $m_{abc} = -m_{bac}$, $m_{abc}\eta^{ac} = 0$, and $m_{abc} + m_{cab} + m_{bca} = 0$.

Then the Cartan equation reads:

$$A_{abc} = -8\pi G\, a_{abc}, \quad V_a = \tfrac{1}{3}\,8\pi G\, s_a, \quad M_{cab} = -8\pi G\, m_{abc}. \tag{49}$$

In the static, spherical case, we have $A_{abc} = 0$,

$$V_t = \tfrac{1}{3}(2\,C_t + D_r)/\sqrt{B}, \qquad V_r = \tfrac{1}{3}(-\tfrac{1}{2}B'/B - 2/r + 2\,C_r + D_t\,A/B)/\sqrt{A}, \tag{50}$$

and

$$M_{rtr} = \tfrac{2}{3}(-C_t + D_r)/\sqrt{B}, \qquad M_{trt} = \tfrac{2}{3}(\tfrac{1}{2}B'/B - 1/r + C_r - D_t\,A/B)/\sqrt{A}, \tag{51}$$

$$M_{\theta t\theta} = M_{\varphi t\varphi} = -\tfrac{1}{2}M_{rtr}, \qquad M_{\theta r\theta} = M_{\varphi r\varphi} = \tfrac{1}{2}M_{trt}. \tag{52}$$

3.4 Second step: vacuum solution

In vacuum, $\tau_{ab} = 0$ and $s_{abc} = 0$, we retrieve the Kottler solution, equation (7). Indeed by Cartan's equation, vanishing spin current implies vanishing torsion: $C_t = D_r = 0$, $C_r = 1/r$ and $D_t = \frac{1}{2}B'/A$ and then the invariant, metric connection (19 - 22) reduces to the (symmetric) Christoffel symbols.

4 Third step: a Schwarzschild star with torsion

Note that metric and connection in the static, spherical case are automatically invariant under space inversion. (This is not true in the homogeneous, isotropic case [4, 6].) However we do not have invariance under time reversal and to be precise we should say 'stationary' rather than 'static'.

To construct a counter example to the weak Gauß law, we do suppose invariance under time reversal. This is certainly not justified for our sun, but not unreasonable for a Schwarzschild star with constant mass density $\mathrm{d}\rho/\mathrm{d}r = 0$ inside the radius R. The following functions are odd under time reversal and must vanish: all connection, curvature, Ricci, energy-momentum, torsion and spin tensor components with an odd number of indices equal to t. Consequently C_t, D_r, q, s_t and m_{rtr} are zero and we remain with four unknown functions of r in the fields: B, A, C_r and D_t. In the sources we still have five arbitrary functions of r: the mass density ρ the radial and azimuthal pressure p_r, p_a and the spin densities s_r and m_{trt}. They define the right-hand sides of the five remaining field equations, three Einstein and two Cartan equations.

To continue, we set $D_t = \frac{1}{2}B'/A$ and simplify notations $C := C_r$, $s := s_r$. Then the two Cartan equations reduce to:

$$2\,(C - 1/r)/\sqrt{A} = 8\pi G\, s, \quad m_{trt} = \tfrac{1}{3}s. \tag{53}$$

Now we may introduce a Schwarzschild star by assuming that the mass density ρ and the spin density s are constant, i.e. r-independent, with an equation of state: $s = w\rho$.

Upon eliminating C via the Cartan equation, the tt, rr and $\theta\theta$ components (39 - 41) of Einstein's equation reduce to:

$$A'/(rA^2) + 1/r^2 - 1/(r^2A) - 16\pi G\, w\rho/(r\sqrt{A}) - (4\pi G\, w\rho)^2 - \Lambda \;=\; 8\pi G\,\rho, \tag{54}$$

$$\left[1/(rA) + 4\pi G\, w\rho/\sqrt{A}\right]\; B'/B \qquad\qquad -1/r^2 + 1/(r^2A) + 8\pi G\, w\rho/(r\sqrt{A}) + (4\pi G\, w\rho)^2 + \Lambda \;=\; 8\pi G\, p_r, \tag{55}$$

$$\tfrac{1}{2}B''/(AB) - \tfrac{1}{4}(A'/A + B'/B)\,B'/(AB) \qquad -\tfrac{1}{2}(A'/A - B'/B)/(rA) + (\tfrac{1}{2}B'/B + 1/r)\,4\pi G w\rho/\sqrt{A} + \Lambda \;=\; 8\pi G\, p_\mathrm{a}. \tag{56}$$

As with zero torsion, the tt component decouples from the other two equations and can be integrated separately. To redo the third step of section 2 for the tt component with torsion, we now need two definitions of mass: an interior mass,

$$M_i := \int_0^R \rho\, 4\pi\tilde{r}^2\,\mathrm{d}\tilde{r} = \tfrac{4}{3}\pi R^3\,\rho, \quad \rho = \tau^{\mu=t}{}_{\nu=t} = \tau^{a=t}{}_{b=t}, \tag{57}$$

and an exterior mass M_e defined by the strength of the gravitational field outside, $r \geq R$,

$$A(r) =: \left[1 - \frac{2G\,M_e}{r} - \tfrac{1}{3}\Lambda\, r^2\right]^{-1}. \tag{58}$$

In contrast to the torsionless case, they do not coincide:

$$M_e = M_i \left[1 + \frac{6w}{R^3} \int_0^R \frac{\tilde{r}\,\mathrm{d}\tilde{r}}{\sqrt{A(\tilde{r})}} + \frac{3w^2\,GM_i}{2R^3} \right]. \tag{59}$$

Note that for sufficiently large $|w|$ the exterior mass exceeds the interior one, even for negative w. Note also that this mass relation depends on the interior solution $A(r)$ of the tt component of the Einstein equation (54). This solution is not obvious (to us) and we will solve equation (54) numerically. As a test the numerical solution will reproduce for $w = 0$ the Schwarzschild star with cosmological constant [7, 8, 9]. This solution has $p_r = p_\mathrm{a} =: p$. Its functions,

$$A = \frac{1}{W^2}\,, \qquad B = (\alpha K + \beta W)^2, \qquad p = \rho \left[\frac{K}{\alpha K + \beta W} - 1 \right], \tag{60}$$

are continuous at the boundary $r = R$. The auxiliary quantities used are:

$$\gamma := \tfrac{1}{3}(8\pi G\,\rho + \Lambda), \quad \alpha := \tfrac{1}{2}8\pi G\,\rho/\gamma, \quad \beta := (-\tfrac{1}{6}8\pi G\,\rho\, +\, \tfrac{1}{3}\Lambda)/\gamma = 1 - \alpha, \tag{61}$$

$$W(r) := \sqrt{1 - \gamma r^2}, \qquad K := W(R). \tag{62}$$

5 Numerical solution

Equation (54) has an integrable singularity at $r = 0$ which can be avoided by redefining the dependent variable, $a(r) := r/A(r)$, yielding:

$$1 - a' - 16\pi G\,w\rho\sqrt{ra} - (4\pi G\,w\rho\,r)^2 - \Lambda r^2 = 8\pi G\rho\,r^2. \tag{63}$$

We solve this equation by a Runge-Kutta algorithm with initial condition $a(0) = 0$ for $0 \le r \le R$. We check that $a(r)$ remains positive for $0 < r \le R$, and that $\lim_{r\to 0} A(r) = 1$ which also ensures that that $r/\sqrt{A(r)} = \sqrt{r\,a(r)}$ is integrable for $0 \le r \le R$. Then we get the masses from

$$M_i = \tfrac{4}{3}\pi R^3 \rho, \quad M_e = (R - \tfrac{1}{3}\Lambda R^3 - a(R))/(2\,G)\,. \tag{64}$$

For $w = 0$ we reproduce the analytic solution $A = [1 - \frac{1}{3}(8\pi G\,\rho + \Lambda)\,r^2]^{-1}$ and have $M_i = M_e$ in accordance with the weak Gauß law. To obtain a ratio of $M_e/M_i = 5$ for the sun, $M_i = M_\odot$, $R = 7 \cdot 10^8$ m, we must choose $w = 3.1$ s. We get the same ratio for a cluster, $M_i = 10^{15}\,M_\odot$, $R = 3 \cdot 10^{23}$ m, with $w = 2 \cdot 10^{15}$ s. In these two cases the positive definite contribution $\frac{3}{2}w^2\,GM_i/R^3$ to the mass ratio amounts to $6 \cdot 10^{-6}$ and 0.6 respectively. We have used the experimentally favoured value of $\Lambda = 1.5 \cdot 10^{-52}$ m^{-2}. Setting the cosmological constant to zero however does not change the values of M_e/M_i by more than 10^{-5}.

6 Conclusion

A torsion induced failure of the weak Gauß law might be welcome with respect to the dark matter problems. Indeed we have already seen [4] that the Hubble diagram of super novae can be fitted by the Einstein-Cartan theory with $w = 10^{17}$ s and no dark matter. This w-value is not far from the one found here for a spherical cluster. However they are far, far away from the naive microscopic value:

$$w = \frac{\hbar/2}{m_{\mathrm{proton}} c^2} \sim 10^{-25} \text{ s.} \tag{65}$$

It would nevertheless be interesting to compute the rotation curve of a realistic galaxy and lensing in the Einstein-Cartan theory. Both are formidable theoretical challenges.

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